

PAPER • OPEN ACCESS

Ground state entanglement entropy for discrete-time two coupled harmonic oscillators

To cite this article: W Kantayasakun *et al* 2017 *J. Phys.: Conf. Ser.* **901** 012168

View the [article online](#) for updates and enhancements.

You may also like

- [Quantum circuit dynamics via path integrals: Is there a classical action for discrete-time paths?](#)
Mark D Penney, Dax Enshan Koh and Robert W Spekkens
- [Improving autonomous thermal entanglement generation using a common reservoir](#)
Zhong-Xiao Man, Armin Tavakoli, Jonatan Bohr Brask et al.
- [Entanglement entropy of excited states in the quantum Lifshitz model](#)
Juanfernando Angel-Ramelli



The Electrochemical Society
Advancing solid state & electrochemical science & technology

242nd ECS Meeting

Oct 9 – 13, 2022 • Atlanta, GA, US

Early hotel & registration pricing ends September 12

Presenting more than 2,400 technical abstracts in 50 symposia

The meeting for industry & researchers in

BATTERIES
ENERGY TECHNOLOGY
SENSORS AND MORE!

 Register now!

 **ECS Plenary Lecture featuring M. Stanley Whittingham,**
Binghamton University
Nobel Laureate –
2019 Nobel Prize in Chemistry



Ground state entanglement entropy for discrete-time two coupled harmonic oscillators

W Kantayasakun^{1,*}, S Yoo-Kong^{1,2,3}, T Deesuwan⁴, M Tanasittikosol¹
and W Liewrian^{1,2}

¹ Theoretical and Computational Physics (TCP) Group, Department of Physics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand.

² Theoretical and Computational Science Centre (TaCs), Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand.

³ Ratchaburi Campus, King Mongkut's University of Technology Thonburi, Ratchaburi, 70510, Thailand.

⁴ Learning Institute, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand.

*E-mail: moontown_lop@hotmail.com

Abstract. The ground state entanglement of the system, both in discrete-time and continuous-time cases, is quantified through the linear entropy. The result shows that the entanglement increases as the interaction between the particles increases in both time scales. It is also found that the strength of the harmonic potential affects the formation rate of the entanglement of the system. The different feature of the entanglement between continuous-time and discrete-time scales is that, for discrete-time entanglement, there is a cut-off condition. This condition implies that the system can never be in a maximally entangled state.

1. Introduction

The idea that time flow constitutes from discrete-steps was suggested by many physicists [1, 2, 3, 4]. The question which could be raised from this idea is whether or not there are similarities or differences of the physical behaviors at the discrete-time scale and continuous-time scale. To answer this question, the system of two coupled harmonic oscillators is used as a toy model to study at the quantum level. The comparison between the discrete-time wave function and the continuous-time wave function is investigated. Furthermore, an important feature in quantum mechanics called entanglement is examined in detail. What we expect to observe in this study are some extra-features arising due to the discreteness of the time flow.

The organisation of this article is as follows. In Section 2, the formulation of the equations of motion of the two coupled oscillators is set up in both discrete-time and continuous-time scales. Then, in Section 3, the discrete-time wave function is computed and with the modified uncertainty principle. Once the wave function is obtained, the linear entanglement entropy is computed in Section 4 together with the discussion. Finally, the conclusion is provided with some remarks.



2. Discrete-time coupled harmonic oscillators

The system consists of two identical particles with unit mass and the interactions between themselves, and, between the particles and the environment are modelled by Hooke's law with coupling constants σ and k , respectively. The Hamiltonian of the system is given by

$$H(p_1, p_2, x_1, x_2) = p_1^2/2 + p_2^2/2 + kx_1^2/2 + kx_2^2/2 + \sigma(x_1 - x_2)^2/2, \quad (2.1)$$

where p_i and x_i are the momentum and position of the i^{th} particle and $i = 1, 2$. To decouple the Hamiltonian, the normal coordinates $X_1 = (x_1 + x_2)/\sqrt{2}$ (mode 1) and $X_2 = (x_1 - x_2)/\sqrt{2}$ (mode 2) are used to transform Eq. (2.1) into

$$H(P_1, P_2, X_1, X_2) = (P_1^2 + \omega_1^2 X_1^2)/2 + (P_2^2 + \omega_2^2 X_2^2)/2, \quad (2.2)$$

where P_i are new momentum variables and the angular frequencies are $\omega_1 = \sqrt{k}$ (mode 1) and $\omega_2 = \sqrt{(k/2 + \sigma)2}$ (mode 2). We now introduce the discrete-time Hamiltonian [5, 6] given by

$$H(\tilde{P}_1, \tilde{P}_2, X_1, X_2) = (\tilde{P}_1^2 + \omega_1^2 X_1^2)/2 + (\tilde{P}_2^2 + \omega_2^2 X_2^2)/2, \quad (2.3)$$

where $P_i(n)$ and $X_i(n)$ are the discrete-time momentum and position of the i^{th} particle at time n . The shifted momentum is $\tilde{P}_i = P_i(n + \epsilon)$, where ϵ is the discrete-time step. The discrete-time Hamilton equations are $\partial H/\partial \tilde{P}_i = (\tilde{X}_i - X_i)/\epsilon$ and $\partial H/\partial X_i = -(\tilde{P}_i - P_i)/\epsilon$, resulting in discrete maps

$$\tilde{X}_i = (1 - \omega_i^2 \epsilon^2)X_i + P_i \epsilon, \quad \text{and} \quad \tilde{P}_i = -\omega_i^2 \epsilon X_i + P_i. \quad (2.4)$$

From the first equation of (2.4), we find that $P_i = [\tilde{X}_i - (1 - \omega_i^2 \epsilon^2)X_i]/\epsilon$ then $\tilde{P}_i = [\tilde{X}_i - (1 - \omega_i^2 \epsilon^2)\tilde{X}_i]/\epsilon$. Inserting these two equations into the second equation of (2.4), we obtain $\tilde{X}_i + X_i = 2(1 - \omega_i^2 \epsilon^2/2)\tilde{X}_i$ and therefore $\tilde{X}_i + X_i = 2(1 - \omega_i^2 \epsilon^2/2)X_i$ which is the discrete-time equation of motion of the system. Note that, under the continuum limit $\epsilon \rightarrow 0$, the continuous-time equation of motion for the system is recovered.

3. Discrete-time wave function

To obtain the discrete-time wave function, we start with the function [8]

$$\hat{I}_i = \hat{P}_i^2 + \omega_i^2 \hat{X}_i^2 - \epsilon \omega_i^2 (\hat{P}_i \hat{X}_i + \hat{X}_i \hat{P}_i)/2, \quad (3.5)$$

where \hat{P}_i and \hat{X}_i are operators. Since \hat{I}_i is invariant under the map (2.4): $\tilde{\hat{I}}_i = \hat{I}_i$, then it can be treated as the effective Hamiltonian resulting in

$$\hat{I}\Psi(X_1, X_2) = \left[\sum_{i=1}^2 \left(-\hbar^2 \frac{\partial^2}{\partial X_i^2} + i\epsilon \omega_i^2 \hbar X_i \frac{\partial}{\partial X_i} + \omega_i^2 X_i^2 + \frac{i\hbar}{2} \epsilon \omega_i^2 \right) \right] \Psi(X_1, X_2) = E\Psi(X_1, X_2), \quad (3.6)$$

where $\hat{I} = \hat{I}_1 + \hat{I}_2$ is the total effective Hamiltonian operator and E is the total energy of the system. Writing the wave function as $\Psi(X_1, X_2) = \psi(X_1)\varphi(X_2)$ and using the transformations $\varphi(X_1) = w(X_1)\exp[i\epsilon\omega_1^2 X_1^2/(4\hbar)]$ and $\varphi(X_2) = w(X_2)\exp[i\epsilon\omega_2^2 X_2^2/(4\hbar)]$, the eigenfunctions of \hat{I} are

$$\Psi_{nm} = \left(\frac{\Omega_1}{\pi\hbar} \right)^{\frac{1}{4}} \left(\frac{\Omega_2}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \frac{1}{\sqrt{2^m m!}} H_n \left(\sqrt{\frac{\Omega_1}{\hbar}} X_1 \right) H_m \left(\sqrt{\frac{\Omega_2}{\hbar}} X_2 \right) e^{i\left(\frac{\epsilon\omega_1^2}{4\hbar} - \frac{\Omega_1}{2\hbar}\right)X_1^2 + i\left(\frac{\epsilon\omega_2^2}{4\hbar} - \frac{\Omega_2}{2\hbar}\right)X_2^2} \quad (3.7)$$

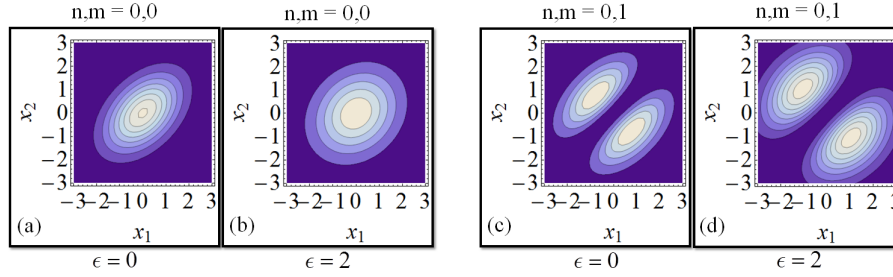


Figure 1. Contour plots of the probability density for ground state (a),(b) and first excited state (c),(d) for $k = 0.1$ and $\sigma = 0.3$.

where $n, m = 0, 1, 2, 3, \dots$ and $H_y(x)$ is the Hermite Polynomial of order y . The total energy now is $E_{nm} = E_n + E_m$, where $E_n = 2\hbar\Omega_1(n + 1/2)$ and $\Omega_1 = \omega_1\sqrt{(1 - \epsilon^2\omega_1^2/4)}$ (mode 1), and, $E_m = 2\hbar\Omega_2(m + 1/2)$ and $\Omega_2 = \omega_2\sqrt{(1 - \epsilon^2\omega_2^2/4)}$ (mode 2). Under the continuum limit $\epsilon \rightarrow 0$, the wave function (3.7) is identical to that of the continuous-time harmonic oscillators. The contour of the probability density is shown in Fig. 1 for the case of $\epsilon = 0$ (continuous-time case), and $\epsilon = 2$. According to Fig. 1, the probability of the discrete-time wave function is a little bit broader than that of the continuous-time wave function. This results from the fact that both the exponential terms $\exp[-\Omega_i X_i^2/(2\hbar)]$ and the Hermite Polynomials $H_y(\sqrt{\Omega_i/\hbar}X_i)$ contain the discrete-time parameter ϵ . Furthermore, we find that the uncertainty principle for each mode in this discrete-time setting is altered to

$$\sigma_{X_i}\sigma_{P_i} = \hbar(y_i + 1/2)\sqrt{1 + \epsilon^2\omega_i^4/4\Omega_i^2}, \quad (3.8)$$

where $y_1 = n$ and $y_2 = m$. This leads to the modified Heisenberg algebra $[X_i, P_i] = i\hbar\sqrt{1 + \epsilon^2\omega_i^4/4\Omega_i^2}$.

4. Entanglement entropy of the ground state

To study the entanglement behavior of the ground state of quantum discrete-time coupled harmonic oscillators, we use the linear-entropy S_L given by

$$S_{L_j} = 1 - \text{Tr}(\rho_j^2), \quad (4.9)$$

where $j = 1, 2$, $\rho_j = \text{Tr}_j \rho_{12} = \int \rho_{12} dx_j$ is the reduced density matrix of the system j , and ρ_{12} is the full density matrix. Note that, for a global pure state, the linear-entropy of the reduced state is bounded between $0 \leq S_L \leq 1$, where $S_L = 1$ indicates the whole system is maximally entangled and $S_L = 0$ indicates a separable state. The full density matrix of the ground state is $\rho_{12}(x_1, x_2; x'_1, x'_2) = \Psi_{00}(x_1, x_2)\Psi_{00}^*(x'_1, x'_2)$ and therefore

$$S_L = 1 - \frac{\gamma - \beta}{\sqrt{\gamma^2 - \beta^2}}, \quad (4.10)$$

where

$$\gamma = \frac{1}{4}(\Omega_1 + \Omega_2) + \frac{\Omega_1\Omega_2}{\Omega_1 + \Omega_2} + \frac{\epsilon^2\sigma^2}{4(\Omega_1 + \Omega_2)}, \beta = \frac{1}{4}(\Omega_1 + \Omega_2) - \frac{\Omega_1\Omega_2}{\Omega_1 + \Omega_2} + \frac{\epsilon^2\sigma^2}{4(\Omega_1 + \Omega_2)}. \quad (4.11)$$

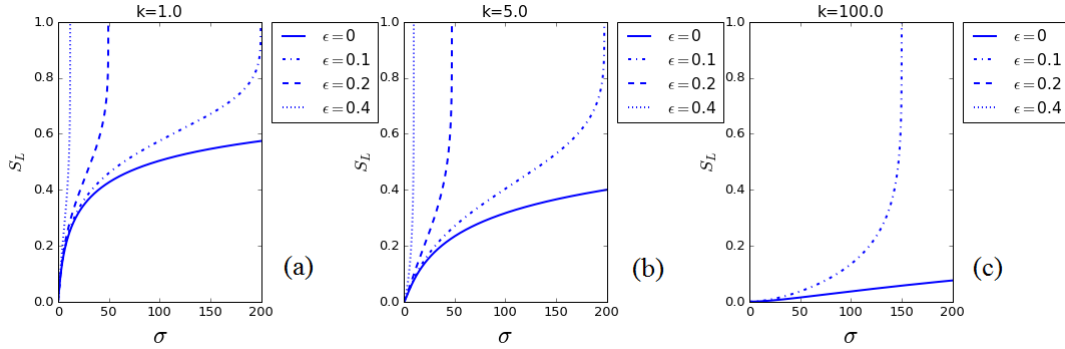


Figure 2. The relation between the linear entropy and the internal interaction (σ) with different amount of the discrete-time scale and the external interaction (k).

According to Fig. 2, in the continuous time (solid lines), the entanglement of the system at the ground state increases as the interaction between particles σ increases, while the interaction with environment k is fixed. The system approaches to the maximally entangled state $S_L \rightarrow 1$ as the parameter σ approaches to infinity implying that the oscillation mode Ω_1 (the center of mass motion) significantly dominates over the oscillation mode Ω_2 (the relative motion). We also find that when the parameter k increases, the entanglement will rise more slowly with the increasing value of the parameter σ . This means that the oscillation mode Ω_2 becomes more significant with increasing k which then makes the oscillation mode Ω_1 more difficult to overcome the oscillation mode Ω_2 . In the case that the parameter k is infinitely large, the entanglement of the system is extremely suppressed due to the domination of the oscillation mode Ω_2 . We may now say that less relative motion (the oscillation mode 2) of the system implies more entanglement.

In the discrete time case, the entanglement of the system behaves almost the same with the continuous time case. Except that we cannot freely vary the values of the parameter σ and the parameter k since there are the cut-off conditions coming from the fact that both Ω_1 and Ω_2 must be positive values. This implies that $0 \leq \omega_2^2 < 4/\epsilon^2$ since $\omega_2 \geq \omega_1$. In terms of σ , this will give the inequality $0 \leq \sigma < 2/\epsilon^2 - k/2$ which also implies that $0 \leq k < 4/\epsilon^2$. Both k and σ cannot satisfy their respective upper bounds ($k = 4/\epsilon^2$ and $\sigma = 2/\epsilon^2 - k/2$) because that will cause the wave function (3.7) to vanish which means the state does not exist (implying that the motion of the system cannot be in any oscillation modes). If $k > 4/\epsilon^2$ (which implies $\sigma > 2/\epsilon^2 - k/2$) the oscillation frequencies Ω_1 and Ω_2 will become imaginary and the wave function is now not well define. This is the reason that $k \geq 4/\epsilon^2$ and $\sigma \geq 2/\epsilon^2 - k/2$ present unphysical situations and have to be excluded from the our consideration. In the physical situations, if we fix the value of the parameter k , the entanglement of the system will increase as the parameter σ increases and the entanglement will only asymptotically approach 1, but never reaches 1, before the parameter σ gets to the cut-off point $\sigma = 2/\epsilon^2 - k/2$. Increasing the value of the parameter k will suppress the entanglement of the system like those in the continuous time case.

5. Concluding discussion

We can analyse and conclude these results from two different perspectives.

Firstly, if we take the view that the discreteness of time is a fundamental property of the universe, we find that the discrete-time flow affects the system behaviors. Some extra-features, e.g. broader probability contour, modification of the uncertainty principle and cut-off conditions for the ground state entanglement entropy, naturally showed up and will be washed away under the continuum limit [7]. Interestingly, we find an unexpected relationship between the discrete-

time step ϵ , the strength of the mutual interaction between the two subsystems σ , and the strength of the harmonic potential k . In particular, we find that σ is bounded from above by a function of k and k is also bounded from above by a function of ϵ . This behavior is completely different from the continuous-time scale ($\epsilon = 0$), where the values of k and σ are totally independent.

Secondly, if we look at these results from the operational point of view. By assuming that time is fundamentally continuous but treating that the discreteness arises from experimental samplings of the positions and velocities of the system at a given frequency determined by $1/\epsilon$, we interpret that the difference in the linear entropy for each value of ϵ is due to the difference in the sampling rate itself. We also discover that the bounds are not actually physical but appear due to the fact that the corresponding cut-off sampling rate ($1/\epsilon_{cut-off} = \sqrt{(\sigma/2) + (k/4)}$) is equal to the Nyquist sampling rate of the system. Thus the reason the observation becomes unphysical beyond that bound is because the sampling rate is less than the Nyquist frequency, which can potentially make the results of the observation become distorted.

Acknowledgments

This work is supported by the Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation Cluster (CLASSIC), Faculty of Science, KMUTT. WK would like to thank Dr.Thana Sutthibutpong for his help on numerical computation and also to Dr.Ekkarat Pongophas for his helpful discussion. SK is supported by National Research Council of Thailand (NRCT) under grant No. 219700.

6. Reference

- [1] A. Einstein, Physics and reality, reprinted in *Essays in Physics*, Philosophical Library, New York, 1936.
- [2] R.P. Feynman, Simulating physics with computers, *International Journal of theoretical Physic* **21** (1982), no.6/7, 467-488.
- [3] T.D. Lee, Can time be a discrete dynamical variable?, *Physics Letters B* **122** (1983), 217-220.
- [4] G.'t Hooft, Quantization of discrete deterministic theories by Hilbert space extension, *Nuclear Physics B* **342** (1996), 471-485.
- [5] C.M. Bender, K.A. Milton, D.H. Sharp, L.M. simmons Jr., and R. stong, Discrete-time quantum mechanics, *Physical Review D* **32** (1985), no.6,1476-1485.
- [6] M.V. Berry, N.L. Balazs, M. Tabor, and A. Vorus, Quantum maps, *Annals of Physics* **122** (1979), 26-63.
- [7] M. Srednicki, Entropy and Area, *Physical Review Letters* **71** (1993), no.5, 666-667.
- [8] C.M. Field, On the Quantization of Integrable Discrete-Time Systems. Ph.D. thesis, University of Leeds, 2005.